# THE QUADRATIC FORMULA – IRRATIONAL SOLUTIONS

## ❑ *INTRODUCTION*


Your first experience with the Quadratic Formula probably involved solving a quadratic equation like $6x^2 + 31x + 40 = 0$, whose solutions are $-\frac{5}{2}$ and $-\frac{8}{3}$. In fact, all of the answers back then were either positive or negative whole numbers, or fractions, or zero. [In other words, all the solutions were *rational numbers*.]

You may have realized that we obtained these kinds of answers because the square-root part of the Quadratic Formula $\left(\sqrt{b^2-4ac}\right)$ always worked out to the square root of a perfect square, like $\sqrt{49}$ or $\sqrt{0}$.

But we know that there are square roots like $\sqrt{5}$. What would we do if we came up with a radical like that, assuming that no calculator is allowed? In this section, we'll see that there's really nothing special to worry about — just work the Quadratic Formula:

$$\text{If } ax^2 + bx + c = 0, \text{ then } x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

EXAMPLE 1: **Solve for $x$: $2x^2 - 3x - 1 = 0$**

Solution: For this quadratic equation, $a = 2$, $b = -3$, and $c = -1$. Inserting these values into the Quadratic Formula gives

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(2)(-1)}}{2(2)}$$

$$= \frac{3 \pm \sqrt{9+8}}{4} = \boxed{\frac{3 \pm \sqrt{17}}{4}}$$

And that's it, because $\sqrt{17}$ cannot be simplified and the fraction cannot be reduced. If you'd like, the two solutions of the equation can be written separately:

$$x = \frac{3+\sqrt{17}}{4}, \ \frac{3-\sqrt{17}}{4}$$

# Homework

1. Solve each quadratic equation:

a. $x^2 - 5x + 1 = 0$

b. $2n^2 + n - 7 = 0$

c. $3a^2 - 3a - 1 = 0$

d. $5t^2 + 7t + 1 = 0$

e. $5x^2 + 15x - 2 = 0$

f. $t^2 + t - 8 = 0$

g. $5h^2 - 13h = -5$

h. $7u^2 + u = 3$

i. $2k^2 + 7k = 8$

j. $2m^2 - 3 = -9m$

2. Solve for $x$: $-7x^2 + 5x + 1 = 0$

Hint: Although not necessary, it's traditional to always make $a$, the leading coefficient, positive. This way, your solutions will more likely match those provided by your teacher or the textbook. Since $a = -7$ in this quadratic equation, we can multiply (or divide) each side of the equation by $-1$, yielding $7x^2 - 5x - 1 = 0$. Now you do the rest.

## ❑ *MORE QUADRATIC EQUATIONS*

<u>EXAMPLE 2:</u> **Solve for $x$: $2x^2 - 8x + 3 = 0$**

<u>Solution:</u> The values of $a$, $b$, and $c$ for use in the Quadratic Formula are

$$a = 2 \qquad b = -8 \qquad c = 3$$

So now we state the Quadratic Formula, insert the three values, and then do a lot of arithmetic and simplifying until the final answer is exact (no decimals) and as clean as possible.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \text{(The Quadratic Formula)}$$

$$= \frac{-(\mathbf{-8}) \pm \sqrt{(\mathbf{-8})^2 - 4(\mathbf{2})(\mathbf{3})}}{2(\mathbf{2})} \qquad (a = 2 \quad b = -8 \quad c = 3)$$

$$= \frac{8 \pm \sqrt{64 - 24}}{4} \qquad \text{(square and multiply)}$$

$$= \frac{8 \pm \sqrt{40}}{4} \qquad \text{(finish the arithmetic)}$$

$$= \frac{8 \pm \sqrt{4 \cdot 10}}{4} \qquad \text{(factor the radicand)}$$

$$= \frac{8 \pm 2\sqrt{10}}{4} \qquad \text{(simplify the radical)}$$

$$= \frac{\cancel{2}(4 \pm \sqrt{10})}{\cancel{2} \cdot 2} \qquad \text{(factor the numerator and reduce)}$$

We write our two solutions as

$$\boxed{x = \frac{4 \pm \sqrt{10}}{2}}$$

<u>EXAMPLE 3:</u> **Solve for $y$: $3y^2 = 13$**

<u>Solution:</u> We begin by subtracting 13 from each side of the equation to convert the equation into standard quadratic form:

$$3y^2 - 13 = 0$$

At this point, we see that $a = 3$, $b = 0$, and $c = -13$. Applying the Quadratic Formula:

$$y = \frac{-\mathbf{0} \pm \sqrt{\mathbf{0}^2 - 4(\mathbf{3})(\mathbf{-13})}}{2(\mathbf{3})} = \frac{\pm\sqrt{0+156}}{6} = \frac{\pm\sqrt{156}}{6}$$

$$= \frac{\pm\sqrt{4 \cdot 39}}{6} = \frac{\pm\sqrt{4}\sqrt{39}}{6} = \frac{\pm 2\sqrt{39}}{6} = \frac{\pm 2\sqrt{39}}{2 \cdot 3} = \frac{\pm \cancel{2}\sqrt{39}}{\cancel{2} \cdot 3}$$

And we have our final answer (two of them!) in simplest form:

$$\boxed{y = \pm\frac{\sqrt{39}}{3}}$$

<u>EXAMPLE 5:</u> **Solve for $n$: $2n^2 = 3n - 4$**

<u>Solution:</u> If we convert to standard form, the equation becomes

$$2n^2 - 3n + 4 = 0$$

where $a = 2$, $b = -3$, and $c = 4$. The Quadratic Formula produces

$$n = \frac{-(\mathbf{-3}) \pm \sqrt{(\mathbf{-3})^2 - 4(\mathbf{2})(\mathbf{4})}}{2(\mathbf{2})} = \frac{3 \pm \sqrt{9-32}}{4} = \frac{3 \pm \sqrt{-23}}{4}$$

Remembering that $\sqrt{-23}$ is not a number in this Algebra course, we stop right here and declare that the given equation has

$$\boxed{\text{No Solution}}$$

# Homework

3. Solve each quadratic equation:

a. $3x^2 - 6x - 1 = 0$ b. $2k^2 = 8k - 6$

c. $2n^2 - 10 = 0$ d. $y^2 = 49$

e. $14d^2 - 4d = 4$ f. $x^2 + 3x + 10 = 0$

g. $5a^2 + 7a - 2 = 0$ h. $u^2 = 40$

i. $3z^2 = 3z$ j. $5x^2 - x + 1 = 0$

k. $x(x - 3) = 7$ l. $y(y + 1) = y(y - 7) + 13$

4. Solve each quadratic equation:

a. $x^2 + 6x + 1 = 0$ b. $2y^2 + 7y = 2$

c. $-n^2 + 28 = 0$ d. $5d^2 = 7d + 1$

e. $x^2 = -4 - 5x$ f. $x^2 + 2x + 1 = 0$

g. $-3a^2 = 5 - 12a$ h. $2x^2 = 7$

i. $3z^2 = 12z$ j. $n^2 + 9 = 0$

k. $-4n^2 + 12n = 9$ l. $t^2 - 9 = 0$

# Review Problems

5. Solve each quadratic equation:

a. $x^2 - x - 56 = 0$

b. $u^2 - 8u = 0$

c. $r^2 = 9$

d. $9k^2 + 49 = -42k$

e. $n^2 + 2n - 80 = 0$

f. $-3t^2 = 11t + 6$

g. $2m^2 - 7m = -6$

h. $8d^2 = 2$

i. $-3a^2 = 7a - 6$

j. $-8h^2 - 8h = 0$

k. $n^2 - 3n + 4 = 0$

l. $2p^2 - 3p - 5 = 0$

m. $-3m^2 - 8m + 3 = 0$

n. $-8z^2 + 2z = 0$

o. $-3w^2 + 3 = 0$

p. $5g^2 = 5g + 2$

q. $-3y^2 + 6y = -3$

r. $12t^2 - 8t - 16 = 0$

s. $3x^2 = -15x + 3$

t. $0 = 21g^2 + 9g - 15$

u. $-5z^2 + 4z = -7$

v. $8q^2 + 4q = 3$

w. $y^2 - 4y = -2$

x. $2x^2 - 2x - 3 = 0$

y. $0 = 14d^2 - 4d - 4$

z. $27x^2 - 6x - 15 = 0$

# Solutions

**1.** a. $\dfrac{5 \pm \sqrt{21}}{2}$ b. $\dfrac{-1 \pm \sqrt{57}}{4}$ c. $\dfrac{3 \pm \sqrt{21}}{6}$ d. $\dfrac{-7 \pm \sqrt{29}}{10}$

e. $\dfrac{-15 \pm \sqrt{265}}{10}$ f. $\dfrac{-1 \pm \sqrt{33}}{2}$ g. $\dfrac{13 \pm \sqrt{69}}{10}$ h. $\dfrac{-1 \pm \sqrt{85}}{14}$

i. $\dfrac{-7 \pm \sqrt{113}}{4}$ j. $\dfrac{-9 \pm \sqrt{105}}{4}$

**2.** $\dfrac{5 \pm \sqrt{53}}{14}$

**3.** a. The quadratic equation $3x^2 - 6x - 1 = 0$ is in standard form. We can read the values $a = 3$, $b = -6$, and $c = -1$. Placing these values into the Quadratic Formula gives

$$x = \frac{-(-6) \pm \sqrt{(-6)^2 - 4(3)(-1)}}{2(3)} = \frac{6 \pm \sqrt{36+12}}{6} = \frac{6 \pm \sqrt{48}}{6} = \frac{6 \pm \sqrt{16 \cdot 3}}{6}$$

$$= \frac{6 \pm 4\sqrt{3}}{6} = \frac{2(3 \pm 2\sqrt{3})}{2 \cdot 3} = \frac{\cancel{2}(3 \pm 2\sqrt{3})}{\cancel{2} \cdot 3} = \frac{3 \pm 2\sqrt{3}}{3}$$

b. 1, 3 c. $\pm\sqrt{5}$ d. $\pm 7$

e. $\dfrac{1 \pm \sqrt{15}}{7}$ f. No solution g. $\dfrac{-7 \pm \sqrt{89}}{10}$ h. $\pm 2\sqrt{10}$

i. 0, 1 j. No solution k. $\dfrac{3 \pm \sqrt{37}}{2}$ l. $\dfrac{13}{8}$

**4.** a. $-3 \pm 2\sqrt{2}$ b. $\dfrac{-7 \pm \sqrt{65}}{4}$ c. $\pm 2\sqrt{7}$ d. $\dfrac{7 \pm \sqrt{69}}{10}$

e. $-1, -4$ f. $-1$ g. $\dfrac{6 \pm \sqrt{21}}{3}$ h. $\pm\dfrac{\sqrt{14}}{2}$

i. 0, 4 j. No solution k. $\dfrac{3}{2}$ l. $\pm 3$

**5.** a. $-7, 8$ b. $8, 0$ c. $\pm 3$ d. $-\frac{7}{3}$

e. $-10, 8$ f. $-3, -\frac{2}{3}$ g. $2, \frac{3}{2}$ h. $\frac{1}{2}, -\frac{1}{2}$

i. $-3, \frac{2}{3}$ j. $-1, 0$ k. No solution l. $\frac{5}{2}, -1$

m. $-3, \frac{1}{3}$ n. $0, \frac{1}{4}$ o. $\pm 1$ p. $\frac{5 \pm \sqrt{65}}{10}$

q. $1 \pm \sqrt{2}$ r. $\frac{1 \pm \sqrt{13}}{3}$ s. $\frac{-5 \pm \sqrt{29}}{2}$ t. $\frac{-3 \pm \sqrt{149}}{14}$

u. $\frac{2 \pm \sqrt{39}}{5}$ v. $\frac{-1 \pm \sqrt{7}}{4}$ w. $2 \pm \sqrt{2}$ x. $\frac{1 \pm \sqrt{7}}{2}$

y. $\frac{1 \pm \sqrt{15}}{7}$ z. $\frac{1 \pm \sqrt{46}}{9}$

*“Do, or do not.*

*There is no 'try'.”*

– Yoda (*The Empire Strikes Back*)